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LETTER TO THE EDITOR

Linear divergence in the two-dimensional Ising spin–spin correlator and gauge invariance

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Abstract. We consider the linear divergence which appears in field-theoretical computations of the 2D Ising spin-spin correlation function. By expressing the correlator in terms of fermionic determinants, we show that this divergence is connected with a certain gauge transformation of the fermion model. We prove that if gauge invariance is preserved then the divergent term is not present in the final result

Since Onsager first derived the free energy of the 2D Ising model [1], this system remained an outstanding problem in statistical mechanics. Onsager's solution was later shown to be simply explained in terms of a single Majorana fermion which becomes massless at the critical point [2]. This field-theoretical approach was then applied to the computation of the spin-spin correlation function, $\langle\sigma_0\sigma_R\rangle$ [3-5]. In particular, at the critical temperature, Bander and Itzykson [5] obtained

$$\ln\langle\sigma_0\sigma_R\rangle = \frac{1}{8}\int_0^R \mathrm{d}x\,\mathrm{d}y\,\frac{1}{(x-y)^2+a^2} = \frac{1}{8}\ln\frac{a^2}{a^2+R^2}+\frac{R}{4a}\tan^{-1}\frac{R}{a} \tag{1}$$

where a is the lattice constant. After dropping out the linear divergence in R, for $R \gg a$, one is led to the well known critical behaviour:

$$\langle\sigma_0\sigma_R\rangle = \left(\frac{a}{R}\right)^{1/4}. \tag{2}$$

The presence of the linear divergence is not a special feature of this model. Indeed, the same occurs, for instance, when computing the spin-spin correlator of the 2D random-bond Ising model by similar continuum methods [6, 7].

In this letter we study the linear divergence which appears in [1] from a new point of view. By using a recently proposed path-integral approach to the critical behaviour of 2D systems [8], we show that the above-mentioned divergent part of $\langle\sigma_0\sigma_R\rangle$ is closely related to a certain kind of gauge transformation of the fermionic theory employed to compute the correlator. Moreover, in our formulation it will become apparent that such a divergence is not present when gauge symmetry is preserved. To be more precise, let us write the square of the critical correlation function in the form [4]:

$$\langle\sigma_0\sigma_R\rangle^2 = \left\langle \exp\left(\pi\int_0^R \mathrm{d}x_1\, J_0(x_1)\right)\right\rangle \tag{3}$$

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where $J_\mu = \bar\Psi \gamma_\mu \psi$ is the Dirac fermion current and $\langle\ \rangle$ on the right-hand side is the vacuum expectation value in a model of free massless fermion fields. Note that the squaring allows one to work with Dirac fermions instead of the Majorana fermions which arise in the original transfer matrix formalism [2]. (This trick, first introduced by Ferrell [9], has been recently used in the context of the 2D random-bond Ising model [10].)

We shall work in Euclidean 2D spacetime with matrices chosen in the form

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1 .$$

The next step is to express the line integral in (3) as

$$\int_0^R J_0(x_1)\,\mathrm{d}x_1 = \int \mathrm{d}^2x\, \bar\psi \not{A} \psi$$

where we have introduced the auxiliary vector field A_μ with components

$$\begin{aligned} A_0 &= \delta(x_0)\theta(x_1)\theta(R - x_1) \\ A_1 &= 0. \end{aligned} \tag{4}$$

In two dimensions A_μ can always be written as

$$A_\mu = \varepsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta. \tag{5}$$

Due to (4), the classical fields ϕ and η must obey

$$\begin{aligned} \partial_1\phi + \partial_0\eta &= \delta(x_0)\theta(x_1)\theta(R - x_1) \\ -\partial_0\phi + \partial_1\eta &= 0. \end{aligned} \tag{6}$$

Now (3) can be expressed in terms of fermionic determinants:

$$\langle\sigma_0\sigma_R\rangle^2 = \frac{\det(\mathrm{i}\not\partial + \pi\not{A})}{\det(\mathrm{i}\not\partial)}. \tag{7}$$

At this stage we perform the following change of path-integral fermionic variables:

$$\begin{aligned} \psi(x) &= U^{(-)}(x)\chi(x) \\ \bar\psi(x) &= \bar\chi(x)U^{(+)}(x) \\ U^{(\pm)}(x) &= \exp\{-\pi[\gamma_5\phi(x) \pm \mathrm{i}\eta(x)]\} \end{aligned} \tag{8}$$

which has been chosen so as to cancel the coupling between scalars and fermions in the kinetic term of the effective Lagrangian density

$$\mathscr{L}_{\mathrm{eff}} = \bar\chi U^{(+)}(\mathrm{i}\not\partial + \pi\not{A})U^{(-)}\chi \equiv \bar\chi \mathrm{i}\not\partial\chi.$$

Of course, the Fujikawa Jacobian [11] must be taken into account:

$$\mathscr{D}\bar\psi\mathscr{D}\psi = J_{\mathrm{F}}\mathscr{D}\bar\chi\mathscr{D}\chi$$

and the squared correlator thus becomes

$$\langle\sigma_0\sigma_R\rangle^2 = J_{\mathrm{F}}. \tag{9}$$

As is well known, the fermion Jacobian is not trivial because of the non-invariance of the measure under chiral changes. Its computation requires a regularisation prescription. For gauge theories with Dirac fermions one is naturally led to consider a regularisation scheme that preserves gauge invariance. On the other hand, when the vector field A_μ does not represent a gauge field, one can choose a more general regulator [12, 13]. In this case, a Jacobian which depends on an arbitrary parameter is obtained. The Thirring model [14] and the chiral Schwinger model [15] are examples in which regularisation ambiguities take place (for a thorough discussion of these issues see [13]). In our case the singular potential A_μ has been introduced as a computational artefact and therefore, in principle, it is not connected with a physical gauge field. This means that we should regularise the fermionic determinant in (7) with a general regulator (we shall see below how this regularisation ambiguity can be fixed on physical grounds). Following this route we get a Jacobian with a covariant counterterm in the form

$$\ln J_{\mathrm{F}} = -\frac{\pi}{2}\int \mathrm{d}^2x\, \phi\varepsilon_{\mu\nu}\partial_\mu A_\nu + \frac{b}{2}\int \mathrm{d}^2x\, A_\mu^2 \tag{10}$$

where b is an arbitrary parameter which can be determined only on gauge invariance grounds [13].

Now using (5), (6) and (9) we obtain

$$\ln\langle\sigma_0\sigma_R\rangle = \frac{b-\pi}{2}[\phi(R)-\phi(0)] + \frac{b}{4}\int_0^R \mathrm{d}x_1 \left.\frac{\partial\eta}{\partial x_0}\right|_{x_0=0}. \tag{11}$$

From (6) it follows that

$$\Box\phi(x_0, x_1) = \delta(x_0)\frac{\mathrm{d}}{\mathrm{d}x_1}[\theta(x_1)\theta(R-x_1)]$$

$$\Box\eta(x_0, x_1) = \theta(x_1)\theta(R-x_1)\frac{\mathrm{d}}{\mathrm{d}x_0}[\delta(x_0)]$$

which can be easily solved by considering the convolutions

$$\phi(x) = \int \mathrm{d}^2x'\, G(x,x')\delta(x_0')\frac{\mathrm{d}}{\mathrm{d}x_1'}[\theta(x_1')\theta(R-x_1')]$$

$$\eta(x) = \int \mathrm{d}^2x'\, G(x,x')\theta(x_1')\theta(R-x_1')\frac{\mathrm{d}}{\mathrm{d}x_0'}[\delta(x_0')]$$

where $G(x, x')$ is the Green function of the problem $\Box G(x, x') = \delta^2(x-x')$, given by $G(x, x') = 1/4\pi \ln(|x-x'|^2 + a^2)$, and a is an ultraviolet cut-off provided by the lattice spacing. We then get

$$\phi(x) = -\frac{1}{4\pi}\ln\left(\frac{x_0^2+(x_1-R)^2+a^2}{x_0^2+x_1^2+a^2}\right)$$

and

$$\eta(x) = \frac{1}{2\pi}\int_0^R \mathrm{d}x_1' \frac{x_0}{x_0^2+(x_1'-x_1)^2+a^2}.$$

If these expressions are substituted into (11) we readily obtain

$$\ln\langle\sigma_0\sigma_R\rangle = \frac{\pi-b}{8\pi}\ln\left(\frac{a^2}{a^2+R^2}\right) + \frac{b}{8\pi}\int_0^R \frac{\mathrm{d}x_1\,\mathrm{d}x_1'}{(x_1'-x_1)^2+a^2}$$

where one recognises, in the second term, the same double integral as in (1). Its evaluation is straightforward, yielding

$$\int_0^R \frac{dx\, dy}{(x-y)^2+a^2} = \ln\left(\frac{z^2}{a^2+R^2}\right) + \frac{2R}{a}\tan^{-1}\left(\frac{R}{a}\right).$$

The final answer, before taking the limit for $R \gg a$, is then given by

$$\ln\langle\sigma_0\sigma_R\rangle = \frac{1}{8\pi}\ln\left(\frac{a^2}{a^2+R^2}\right) + \frac{b}{4\pi}\frac{R}{a}\tan^{-1}\left(\frac{R}{a}\right). \tag{12}$$

Therefore, once again we see the appearance of the linear divergence for large R in agreement with the authors of reference [5] who followed a different method. The main point here is that, in our approach, the divergent term occurs as a consequence of the regularisation procedure. Furthermore, it will not be present if we set $b=0$, which corresponds to a gauge invariant regularisation. Thus, in this new formulation one does not need to disregard the undesired term, but one can rather invoke persistence of the formal gauge symmetry of the fermion model under quantum fluctuations.

In summary, we have related the linear divergence which appears in field-theoretical computations of the 2D Ising spin-spin correlator to the regularisation ambiguity of a fermionic determinant. Taking advantage of a recently introduced path-integral treatment of critical 2D models [8], we showed that no divergence takes place when one chooses a regularisation scheme which preserves, at the quantum level, the classical symmetry of the effective Lagrangian

$$\mathscr{L}_{\text{eff}} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ + \pi A\!\!\!/)\psi \tag{13}$$

(with A_μ given by (4)) under the local gauge transformations

$$\begin{aligned} \psi(x) &\to \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x)\,\mathrm{e}^{-\mathrm{i}\alpha(x)} \\ A_\mu(x) &\to A_\mu(x) + \frac{1}{\pi}\partial_\mu\alpha(x). \end{aligned} \tag{14}$$

Note that the auxiliary field A_μ has been introduced in order to represent the line integral of the fermionic current (see (3)). Consequently, demanding gauge invariance under transformation (14) for the effective Lagrangian (13) ensures conservation of the fermionic current for the original model.

It would be interesting to find out if a similar link could be established for critical correlators of other 2D systems. In particular, one could examine the 2D random-bond Ising model where a linear divergence, connected with inhomogeneities in the RG equations, is known to exist [6, 7]. In this case one should deal with a non-Abelian quantum field theory, the $N \to 0$ limit of the O(N) Gross–Neveu model [6]. We hope to report on this problem in a future contribution.

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